Infinitely Logarithmically Monotonic Combinatorial Sequences

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Abstract. Based on the classical concept of continuous logarithmically completely monotonic functions, we introduce the notion of infinitely logarithmically monotonic sequences. By establishing a connection between completely monotonic functions and infinitely logarithmically monotonic sequences, we show that the sequences of the Bernoulli numbers, the Catalan numbers and the central binomial coefficients are infinitely logarithmically monotonic. In particular, a logarithmically monotonic sequence $\{a_n\}_{n\geq 0}$ of order two is ratio log-concave in the sense that $\{a_{n+1}/a_n\}_{n\geq 0}$ is log-concave. We prove that if $\{a_n\}_{n\geq k}$ is ratio log-concave, then $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log-concave under a certain initial condition. We show that sequences of the derangement numbers, the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the numbers of treelike polyhexes and the Domb numbers are ratio log-concave. This implies some known results of Luca and Stănică, and Hou, Sun and Wen on the log-behavior of combinatorial sequences. Moreover, we confirm a conjecture of Sun on the Domb numbers, and we show this property also holds for the Catalan numbers, the central binomial coefficients, the Fine numbers and the numbers of treelike polyhexes.

Keywords: logarithmically completely monotonic function, infinitely logarithmically monotonic sequence, ratio log-concave, Riemann zeta function

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1 Introduction

In this paper, we introduce the notion of infinitely logarithmically monotonic sequences based on the classical concept of logarithmically completely

monotonic functions. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.1}$$

for $x \in I$ and all integers $n \geq 0$.

A positive function f is said to be logarithmically completely monotonic on an interval I if $\log f$ satisfies

$$(-1)^n [\log f(x)]^{(n)} \geq 0 \tag{1.2}$$

for $x \in I$ and all integers $n \geq 1$, see Atanassov and Tsoukrovski [4]. A logarithmically function is completely monotonic, but not vice versa, see Berg [5].

Completely monotonic functions appear naturally in various fields such as probability theory, see Feller [12] and Ismail and Muldoon [16]; numerical analysis, see Frenzen [13] and Wimp [25]; physics, see Day [10]; and potential theory, see Berg and Forst [6]. Many classical functions are completely monotonic, see Alzer and Berg [3] and Miller and Samko [20]. A detailed account of the properties of completely monotonic functions can be found in Widder [24, Chapter IV].

Recall that a sequence $\{a_n\}_{n\geq 0}$ is said to be log-concave (resp. log-convex) if for all $n \geq 1$, $a_n^2 \geq a_{n-1}a_{n+1}$ (resp. $a_n^2 \leq a_{n-1}a_{n+1}$), and it is said to be strictly log-concave (resp. strictly log-convex) if the equality does not hold. Define an operator $\mathcal{R}$ on a sequence $\{a_n\}_{n\geq 0}$ by

$$\mathcal{R}\{a_n\}_{n\geq 0} = \{b_n\}_{n\geq 0},$$

where $b_n = a_{n+1}/a_n$. We say that the sequence $\{a_n\}_{n\geq 0}$ is logarithmically monotonic of order k if for r odd and not greater than $k-1$, the sequence $\mathcal{R}^r\{a_n\}_{n\geq 0}$ is log-concave and for r even and not greater than $k-1$, the sequence $\mathcal{R}^r\{a_n\}_{n\geq 0}$ is log-convex. A sequence $\{a_n\}_{n\geq 0}$ is called infinitely logarithmically monotonic if it is logarithmically monotonic of order k for all integers $k \geq 0$.

We establish a connection between completely monotonic functions and infinitely logarithmically monotonic sequences. Using the log-behavior of the gamma function and the Riemann zeta function, we show that the sequences of the Bernoulli numbers, the Catalan numbers and the central binomial coefficients are infinitely logarithmically monotonic.

A logarithmically monotonic sequence of order two is of special interest. More precisely, a sequence $\{a_n\}_{n\geq 0}$ is logarithmically monotonic sequence of order two if and only if it is log-convex and the ratio sequence

$\{a_{n+1}/a_n\}_{n\geq 0}$ is log-concave. A sequence $\{a_n\}_{n\geq 0}$ is said to be ratio log-concave if $\{a_{n+1}/a_n\}_{n\geq 0}$ is log-concave. Similarly, a sequence $\{a_n\}_{n\geq 0}$ is called ratio log-convex if the ratio sequence $\{a_{n+1}/a_n\}_{n\geq 0}$ is log-convex.

We prove that under a certain initial condition, the ratio log-concavity of a sequence $\{a_n\}_{n\geq k}$ of positive numbers implies that the sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log-concave. Motivated by Firoozbakht's conjecture that the sequence $\{\sqrt[n]{p_n}\}_{n\geq 1}$ is strictly increasing, where p_n is the n-th prime number, Sun [23] conjectured that for some combinatorial sequences $\{a_n\}_{n\geq 0}$, the sequences $\{\sqrt[n]{a_n}\}_{n\geq 1}$ are strictly log-concave except for the first few terms. Hou, Sun and Wen [15] proved this conjecture for the derangement numbers. Luca and Stănică's [19] proved the conjectures for the Bernoulli numbers, the Euler numbers, the Tangent numbers, the Motzkin numbers, the Apécy numbers, the Franel numbers, the central Delannoy numbers and the Schröder numbers.

In this paper, we prove that for the derangement numbers, the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the numbers of treelike polyhexes and the Domb numbers, the sequences $\{a_n\}_{n\geq 0}$ are ratio log-concave except for the first few terms. As consequences, we obtain the results of Hou, Sun and Wen on derangement numbers and the results of Luca and Stănică on the Bernoulli numbers, the Motzkin numbers and the central Delonnay numbers. Moreover, we confirm a conjecture of Sun on the log-behavior of the Domb numbers. We also show that for the Catalan numbers, the central binomial coefficients, the Fine numbers and the numbers of treelike polyhexes, the sequence of $\{\sqrt[n]{a_n}\}_{n\geq 0}$ is strictly log-concave except for the first few terms.

Our approach also applies to the log-behavior of the harmonic numbers. We prove that the sequence of the generalized harmonic numbers $\{H_{n,m}\}_{n\geq 1}$ is ratio log-convex. It implies the strict log-convexity of the sequence $\{\sqrt[n]{H_{n,m}}\}_{n\geq 3}$, which was conjectured by Sun [23] and confirmed by Hou, Sun and Wen [15].

To conclude, we conjecture that the aforementioned combinatorial sequences are almost infinitely logarithmically monotonic in the sense that for each integer $k\geq 0$, these sequences are logarithmically monotonic of order k except for some entries at the beginning.

2 Infinitely Logarithmically Monotonicity

In this section, we first establish a connection between completely monotonic functions and infinitely logarithmically monotonic sequences. By showing that the Riemann zeta function is logarithmically completely monotonic, we

deduce that the sequence of Bernoulli numbers is infinitely logarithmically monotonic. Using the fact that $[\log\Gamma(x)]''$ is completely monotonic, we show that the sequences of Catalan numbers and the central binomial coefficients are infinitely logarithmically monotonic.

Theorem 2.1 *Assume that a function $f(x)$ such that $[\log f(x)]''$ is completely monotonic for $x \geq 1$ and $a_n = f(n)$ for $n \geq 1$. Then the sequence $\{a_n\}_{n\geq 1}$ is infinitely logarithmically monotonic.*

Proof. Denote the sequence $\mathcal{R}^i\{a_n\}_{n\geq 1}$ by $\{a_{n,i}\}_{n\geq 1}$ for all integers $i \geq 0$. In other words,

$$a_{n,0} = a_n \qquad \text{and} \qquad a_{n,i+1} = a_{n+1,i}/a_{n,i}.$$

Consider the functions $f_0(x)$, $f_1(x)$, $f_2(x), \ldots$, where

$$f_0(x) = f(x) \qquad \text{and} \qquad f_{i+1}(x) = f_i(x+1)/f_i(x).$$

Since $a_{n,0} = f_0(n)$, it is obvious that $a_{n,i} = f_i(n)$. Clearly,

$$\log f_{i+1}(x) = \log f_i(x+1) - \log f_i(x),$$

so we have

$$[\log f_{i+1}(x)]^{(k)} = [\log f_i(x+1)]^{(k)} - [\log f_i(x)]^{(k)}. \tag{2.1}$$

On the other hand, since $[\log f(x)]''$ is completely monotonic, we find that for $k \geq 2$,

$$(-1)^k[\log f(x)]^{(k)} \geq 0. \tag{2.2}$$

We aim to show that for $i \geq 0$ and $k \geq 2$,

$$(-1)^k[\log f_{2i}(x)]^{(k)} \geq 0, \tag{2.3}$$

and

$$(-1)^k[\log f_{2i+1}(x)]^{(k)} \leq 0. \tag{2.4}$$

We use induction on i. Clearly, when $i = 0, 1$, (2.3) and (2.4) hold. Assume that for $i \leq n-1$ and $k \geq 2$,

$$(-1)^k[\log f_{2i}(x)]^{(k)} \geq 0, \tag{2.5}$$

and

$$(-1)^k[\log f_{2i+1}(x)]^{(k)} \leq 0. \tag{2.6}$$

By the definition of $f_{2n}(x)$, we see that

$$(-1)^k[\log f_{2n}(x)]^{(k)} = (-1)^k[\log f_{2n-1}(x+1)]^{(k)} - (-1)^k[\log f_{2n-1}(x)]^{(k)}.$$

By the induction hypothesis, we deduce that $(-1)^{k+1}[\log f_{2n-1}(x)]^{(k+1)} \le 0$. Hence when $k \ge 2$ is odd,

$$[\log f_{2n-1}(x+1)]^{(k)} \le [\log f_{2n-1}(x)]^{(k)};$$

and when $k \ge 2$ is even,

$$[\log f_{2n-1}(x+1)]^{(k)} \ge [\log f_{2n-1}(x)]^{(k)}.$$

It follows that (2.3) holds for $i = n$. Similarly, it can be shown that $(-1)^k[\log f_{2n+1}(x)]^{(k)} \le 0$, that is, (2.4) holds for $i = n$. This proves that for all nonnegative integers i, the sequence $\mathcal{R}^{2i}\{a_n\}_{n\ge1}$ is log-convex and the sequence $\mathcal{R}^{2i+1}\{a_n\}_{n\ge1}$ is log-concave. This completes the proof. ∎

It is not difficult to see that the Riemann zeta function

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is logarithmically completely monotonic. It is known that for $\Re(s) > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}, \tag{2.7}$$

where $\Lambda(n)$ is known as the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p, & n = p^m \qquad (p \text{ prime}, \ m \ge 1), \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

see [17, pp.3]. Using the above formula (2.7), we find that, for $x > 1$,

$$(-1)^k[\log \zeta(x)]^{(k)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)(\log n)^{k-1}}{n^x} > 0.$$

It follows that $\zeta(x)$ is logarithmically completely monotonic on $(1, \infty)$. Thus $[\log \zeta(x)]''$ is completely monotonic on $(1, \infty)$.

Before we prove that the sequence of Bernoulli numbers is infinitely logarithmically monotonic, we need consider the log-behavior of the gamma function. It is known that for $x > 0$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=1}^{\infty} \left(\frac{1}{x+m-1} - \frac{1}{m} \right), \tag{2.9}$$

where γ is the Euler constant, see [2]. It follows that

$$\left(\frac{\Gamma'(x)}{\Gamma(x)} \right)^{(k)} = (-1)^{k+1} k! \sum_{m=0}^{\infty} \frac{1}{(x+m)^{k+1}}. \tag{2.10}$$

Therefore we obtain that for $k \geq 2$ and $x \geq 1$, $(-1)^k[\log \Gamma(x)]^{(k)} > 0$, i.e., $[\log \Gamma(x)]''$ is completely monotonic on $(0, \infty)$.

To prove the sequence of the Bernoulli numbers $\{|B_{2n}|\}_{n\geq 1}$ is infinitely logarithmically monotonic, we apply the relationship between the Bernoulli numbers B_n and the Riemann zeta function

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|. \tag{2.11}$$

Since $[\log \Gamma(x)]''$ and $\log \zeta(x)$ are completely monotonic, we obtain the following property of the Bernoulli numbers.

Theorem 2.2 *The sequence $\{|B_{2n}|\}_{n\geq 1}$ is infinitely logarithmically monotonic.*

Proof. Rewrite (2.11) as follows

$$|B_{2n}| = \frac{2\Gamma(2n+1)\zeta(2n)}{(2\pi)^{2n}}.$$

Setting

$$z(x) = \frac{2\zeta(2x)\Gamma(2x+1)}{(2\pi)^{2x}}, \tag{2.12}$$

we get $z(n) = |B_{2n}|$. Since for $x \geq 1$ and $k \geq 2$,

$$[\log z(x)]^{(k)} = 2^k[\log \zeta(2x)]^{(k)} + 2^k[\log \Gamma(2x+1)]^{(k)},$$

we see that for $k \geq 2$, $(-1)^k[\log z(x)]^{(k)} > 0$ on $[1, \infty]$. Thus $[\log z(x)]''$ is completely monotonic on $[1, \infty]$. It follows from Theorem 2.1 that the sequence $\{|B_{2n}|\}_{n\geq 1}$ is infinitely logarithmically monotonic. ∎

Next we consider the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.13}$$

Define

$$c(x) = \frac{\Gamma(2x+1)}{\Gamma(x+1)\Gamma(x+2)},$$

then we have $c(n) = C_n$ for any positive integers n. To show that the sequence $\{C_n\}_{n\geq 1}$ is infinitely logarithmically monotonic, we shall prove that $[\log c(x)]''$ is completely monotonic on $[1, \infty]$.

Theorem 2.3 *The sequence $\{C_n\}_{n\geq 1}$ is infinitely logarithmically monotonic.*

Proof. By the definition (2.13) of $c(x)$, we have that for $x > 0$,

$$[\log c(x)]^{(k)} = [\log \Gamma(2x+1)]^{(k)} - [\log \Gamma(x+1)]^{(k)} - [\log \Gamma(x+2)]^{(k)}.$$

In view of (2.10), we obtain that for $k \geq 2$,

$$(-1)^k [\log \Gamma(x+1)]^{(k)} = (k-1)! \sum_{m=0}^{\infty} \frac{1}{(x+1+m)^k},$$

and

$$(-1)^k [\log \Gamma(x+2)]^{(k)} = (k-1)! \sum_{m=0}^{\infty} \frac{1}{(x+2+m)^k}.$$

On the other hand, we have

$$\begin{aligned}
&(-1)^k [\log \Gamma(2x+1)]^{(k)} \\
&= 2^k (k-1)! \sum_{m=0}^{\infty} \frac{1}{(2x+1+m)^k} \\
&= (k-1)! \sum_{m=0}^{\infty} \frac{1}{(x+1/2+m/2)^k} \\
&= (k-1)! \sum_{i=0}^{\infty} \frac{1}{(x+1/2+(2i)/2)^k} + (k-1)! \sum_{i=0}^{\infty} \frac{1}{(x+1/2+(2i+1)/2)^k} \\
&= (k-1)! \sum_{i=0}^{\infty} \frac{1}{(x+1/2+i)^k} + (k-1)! \sum_{i=0}^{\infty} \frac{1}{(x+i+1)^k}.
\end{aligned}$$

It follows that

$$(-1)^k [\log \Gamma(2x+1)]^{(k)} > (-1)^k [\log \Gamma(x+1)]^{(k)} + (-1)^k [\log \Gamma(x+2)]^{(k)}.$$

This proves that for $x \geq 1$ and $k \geq 2$, $(-1)^k [\log c(x)]^{(k)} \geq 0$. Thus $[\log c(x)]''$ is completely monotonic on $[1, \infty]$. By Theorem 2.1, we conclude that the sequence $\{C_n\}_{n\geq 1}$ is infinitely logarithmically monotonic. ∎

Using a similar argument, it can be shown that the sequence $\{\binom{2n}{n}\}_{n\geq 1}$ is infinitely logarithmically monotonic, where $\binom{2n}{n}$ is known as the n-th central binomial coefficient.

3 Ratio Log-Concavity

From now on, we consider the ratio log-concave sequences. We show that the ratio log-concavity (resp. ratio log-convexity) of a sequence $\{a_n\}_{n\geq k}$ implies

the strict log-concavity (resp. log-convexity) of the sequence of the form $\{\sqrt[n]{a_n}\}_{n\geq k}$ under a certain initial condition. We show that the sequence of the derangement numbers is ratio log-concave and the sequence of the generalized harmonic numbers is ratio log-convex. As consequences, we obtain the results of Hou, Sun and Wen [15] on the log-behavior of the sequences of the derangement numbers and the generalized harmonic numbers.

Theorem 3.1 *If a sequence $\{a_n\}_{n\geq k}$ is ratio log-concave and*

$$\frac{\sqrt[k+1]{a_{k+1}}}{\sqrt[k]{a_k}} > \frac{\sqrt[k+2]{a_{k+2}}}{\sqrt[k+1]{a_{k+1}}}, \tag{3.1}$$

then the sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log-concave.

To prove above theorem, we need the following lemmas.

Lemma 3.2 *If a sequence $\{a_n\}_{n\geq k}$ is ratio log-concave, then for any $k \leq i < j \leq n$, we have*

$$\log\frac{a_{j+1}}{a_j} \geq (j-i)\left(\log\frac{a_{n+1}}{a_n} - \log\frac{a_n}{a_{n-1}}\right) + \log\frac{a_{i+1}}{a_i}. \tag{3.2}$$

Proof. The ratio log-concavity of $\{a_n\}_{n\geq k}$ implies that for $k \leq i \leq n-1$,

$$\left(\frac{a_{i+2}}{a_{i+1}}\right)^2 \geq \frac{a_{i+1}}{a_i}\ \frac{a_{i+3}}{a_{i+2}}.$$

Thus, for $k \leq i \leq n-1$,

$$\log\frac{a_{i+2}}{a_{i+1}} - \log\frac{a_{i+1}}{a_i} \geq \log\frac{a_{i+3}}{a_{i+2}} - \log\frac{a_{i+2}}{a_{i+1}}.$$

It follows that for $k \leq i \leq n-1$,

$$\log\frac{a_{i+2}}{a_{i+1}} - \log\frac{a_{i+1}}{a_i} \geq \log\frac{a_{n+1}}{a_n} - \log\frac{a_n}{a_{n-1}}. \tag{3.3}$$

Since for $j > i$,

$$\log\frac{a_{j+1}}{a_j} = \sum_{l=i}^{j-1}\left(\log\frac{a_{l+2}}{a_{l+1}} - \log\frac{a_{l+1}}{a_l}\right) + \log\frac{a_{i+1}}{a_i},$$

By (3.3), we get (3.2). This completes the proof. ∎

Under condition (3.1) in Theorem, we have the following inequality.

Lemma 3.3 *If a ratio log-concave sequence* $\{a_n\}_{n\geq k}$ *satisfies condition* (3.1) *in Theorem 3.1, then we have for* $n > k$,

$$k \log \frac{a_{k+1}}{a_k} > \frac{k^2+k}{2} \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log a_k. \tag{3.4}$$

Proof. Condition (3.1) can be rewritten as

$$\frac{\log a_{k+1}}{k+1} - \frac{\log a_k}{k} > \frac{\log a_{k+2}}{k+2} - \frac{\log a_{k+1}}{k+1},$$

that is,

$$2(k^2+2k)\log a_{k+1} > (k^2+k)\log a_{k+2} + (k^2+3k+2)\log a_k. \tag{3.5}$$

Expressing (3.5) in terms of logarithms of ratios, we find

$$k \log \frac{a_{k+1}}{a_k} > \frac{k^2+k}{2} \left(\log \frac{a_{k+2}}{a_{k+1}} - \log \frac{a_{k+1}}{a_k} \right) + \log a_k. \tag{3.6}$$

By the ratio log-concavity of $\{a_n\}_{n\geq k}$, we deduce that for $k < n$,

$$\log \frac{a_{k+2}}{a_{k+1}} - \log \frac{a_{k+1}}{a_k} \geq \cdots \geq \log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain (3.4). This completes the proof. ∎

We now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. To establish the strict log-concavity of $\{\sqrt[n]{a_n}\}_{n\geq k}$, we aim to show that for $n > k$,

$$\frac{2\log a_n}{n} > \frac{\log a_{n+1}}{n+1} + \frac{\log a_{n-1}}{n-1}. \tag{3.8}$$

It is easily verified that

$$\begin{aligned}
&\frac{2\log a_n}{n} - \frac{\log a_{n+1}}{n+1} - \frac{\log a_{n-1}}{n-1} \\
&= \frac{1}{n(n+1)(n-1)}\left(2(n^2-1)\log a_n - (n^2-n)\log a_{n+1} - (n^2+n)\log a_{n-1}\right) \\
&= \frac{1}{n^3-n}\left[n\left(\log \frac{a_{n+1}}{a_n} + \log \frac{a_n}{a_{n-1}} \right) - n^2 \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) - 2\log a_n \right].
\end{aligned}$$

We wish to estimate $n \log \frac{a_{n+1}}{a_n}$ and $n \log \frac{a_n}{a_{n-1}}$. We claim that

$$n \log \frac{a_{n+1}}{a_n} > \frac{n(n+1)}{2} \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log a_n. \tag{3.9}$$

and

$$n \log \frac{a_n}{a_{n-1}} > \frac{n(n-1)}{2} \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log a_n. \tag{3.10}$$

Setting $j = n$ in (3.2) of Lemma 3.2, we get that for $k \leq i \leq n$,

$$\log \frac{a_{n+1}}{a_n} \geq (n-i) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log \frac{a_{i+1}}{a_i}. \tag{3.11}$$

Summing (3.11) over i from k to $n-1$ gives

$$(n-k) \log \frac{a_{n+1}}{a_n} \geq \sum_{i=k}^{n-1} (n-i) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \sum_{i=k}^{n-1} \log \frac{a_{i+1}}{a_i}. \tag{3.12}$$

Write $k \log \frac{a_{n+1}}{a_n}$ in the following form

$$k \log \frac{a_{n+1}}{a_n} = k \sum_{j=k+1}^{n} \left(\log \frac{a_{j+1}}{a_j} - \log \frac{a_j}{a_{j-1}} \right) + k \log \frac{a_{k+1}}{a_k}. \tag{3.13}$$

It follows from the ratio log-concavity condition (3.7) that

$$k \sum_{j=k+1}^{n} \left(\log \frac{a_{j+1}}{a_j} - \log \frac{a_j}{a_{j-1}} \right) \geq k(n-k) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right). \tag{3.14}$$

Applying (3.14) and Lemma 3.3 to the right hand side of (3.13), we deduce that

$$\begin{aligned} k \log \frac{a_{n+1}}{a_n} &> \left(k(n-k) + \frac{k^2+k}{2} \right) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log a_k \\ &= \sum_{i=0}^{k-1} (n-i) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log a_k. \end{aligned} \tag{3.15}$$

Combining (3.12) and (3.15), we obtain

$$\begin{aligned} n \log \frac{a_{n+1}}{a_n} &> \sum_{i=0}^{n-1} (n-i) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \sum_{i=k}^{n-1} \log \frac{a_{i+1}}{a_i} + \log a_k \\ &= \frac{n(n+1)}{2} \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log a_n. \end{aligned}$$

This verifies (3.9).

We continue to prove (3.10). Setting $j = n-1$ in (3.2) in Lemma 3.2, we see that for $k \leq i \leq n-1$,

$$\log \frac{a_n}{a_{n-1}} \geq (n-i-1) \left(\log \frac{a_{n+1}}{a_n} - \log \frac{a_n}{a_{n-1}} \right) + \log \frac{a_{i+1}}{a_i}. \tag{3.16}$$

Summing (3.16) over i from k to $n-2$, we get

$$(n-k-1)\log\frac{a_n}{a_{n-1}} \geq \sum_{i=k}^{n-2}(n-i-1)\left(\log\frac{a_{n+1}}{a_n}-\log\frac{a_n}{a_{n-1}}\right)+\sum_{i=k}^{n-2}\log\frac{a_{i+1}}{a_i}. \tag{3.17}$$

Note that

$$k\log\frac{a_n}{a_{n-1}} = k\sum_{j=k+1}^{n-1}\left(\log\frac{a_{j+1}}{a_j}-\log\frac{a_j}{a_{j-1}}\right)+k\log\frac{a_{k+1}}{a_k}. \tag{3.18}$$

Using the ratio log-concavity condition (3.7), we find that

$$k\sum_{j=k+1}^{n-1}\left(\log\frac{a_{j+1}}{a_j}-\log\frac{a_j}{a_{j-1}}\right) \geq k(n-k-1)\left(\log\frac{a_{n+1}}{a_n}-\log\frac{a_n}{a_{n-1}}\right). \tag{3.19}$$

Applying (3.19) and Lemma 3.3 to the righthand of (3.18), we obtain

$$k\log\frac{a_n}{a_{n-1}} > \sum_{i=0}^{k}(n-i-1)\left(\log\frac{a_{n+1}}{a_n}-\log\frac{a_n}{a_{n-1}}\right)+\log a_k. \tag{3.20}$$

Combining (3.17) and (3.20), we have

$$\begin{aligned}(n-1)\log\frac{a_n}{a_{n-1}} &> \sum_{i=0}^{n-2}(n-i-1)\left(\log\frac{a_{n+1}}{a_n}-\log\frac{a_n}{a_{n-1}}\right)+\sum_{i=k}^{n-2}\log\frac{a_{i+1}}{a_i}+\log a_k \\ &= \frac{n(n-1)}{2}\left(\log\frac{a_{n+1}}{a_n}-\log\frac{a_n}{a_{n-1}}\right)+\log a_{n-1}.\end{aligned}$$

This proves (3.10).

Combining the estimates of $n\log\frac{a_{n+1}}{a_n}$ and $n\log\frac{a_n}{a_{n-1}}$ as given in (3.9) and (3.10), we deduce that

$$n\left(\log\frac{a_{n+1}}{a_n}+\log\frac{a_n}{a_n-1}\right) > n^2\left(\log\frac{a_{n+1}}{a_n}-\log\frac{a_n}{a_{n-1}}\right)+2\log a_n,$$

which simplifies to

$$\frac{2\log a_n}{n} > \frac{\log a_{n+1}}{n+1}+\frac{\log a_{n-1}}{n-1}.$$

This completes the proof. ∎

In Section 2, we have proved that the sequences $\{|B_{2n}|\}_{n\geq 1}$, $\{C_n\}_{n\geq 1}$ and $\{\binom{2n}{n}\}_{n\geq 1}$ are infinitely logarithmicall monotonic. Hence they are ratio log-concave. By verifying the first few terms of the above sequences, we get the following corollary.

Corollary 3.4 *The sequence* $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 2}$, $\{\sqrt[n]{C_n}\}_{n\geq 1}$ *and* $\{\sqrt[n]{\binom{2n}{n}}\}_{n\geq 1}$ *are strictly log-concave.*

Notice that the log-concavity of the sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geq 2}$ was conjectured by Sun [23] and confirmed by Luca and Stănică [19].

Next we consider the sequence of derangement numbers. For positive integers $n \geq 1$, the n-th derangement number D_n is the number of permutations σ of $\{1, 2, \ldots, n\}$ that have no fixed points, that is, $\sigma(i) \neq i$ for any $i = 1, 2, \ldots, n$. It is known that the sequence $\{D_n\}_{n\geq 2}$ is log-convex, see Liu and Wang [18]. Sun [23] conjectured that the sequence $\{\sqrt[n]{D_n}\}_{n\geq 3}$ is strictly log-concave, which has been confirmed by Hou, Sun and Wen [15]. Now we show that the sequence $\{D_n\}_{n\geq 2}$ is ratio log-concave.

Theorem 3.5 *The sequence* $\{D_n\}_{n\geq 2}$ *is ratio log-concave.*

Proof. To prove that $\{D_n\}_{n\geq 2}$ is ratio log-concave, we proceed to verify that

$$\frac{D_{n+2}^2}{D_{n+1}^2} > \frac{D_{n+3}}{D_{n+2}}\,\frac{D_{n+1}}{D_n}. \tag{3.21}$$

Using the recurrence relation

$$D_n = nD_{n-1} + (-1)^n, \tag{3.22}$$

we get

$$\begin{aligned}
&\frac{D_{n+2}^2}{D_{n+1}^2} - \frac{D_{n+3}}{D_{n+2}}\frac{D_{n+1}}{D_n} &(3.23)\\
&= \left(n+2+\frac{(-1)^{n+2}}{D_{n+1}}\right)^2 - \left(n+3+\frac{(-1)^{n+3}}{D_{n+2}}\right)\left(n+1+\frac{(-1)^{n+1}}{D_n}\right)\\
&\geq 1 - \frac{n+1}{D_{n+2}} - \frac{n+3}{D_n} - \frac{2(n+2)}{D_{n+1}} + \frac{1}{D_{n+1}^2} - \frac{1}{D_{n+2}D_n}. &(3.24)
\end{aligned}$$

From (3.22), it is easily seen that for $n \geq 5$,

$$D_n > 5(n+3). \tag{3.25}$$

It follows that

$$\frac{n+1}{D_{n+2}} + \frac{n+3}{D_n} + \frac{2(n+2)}{D_{n+1}} < \frac{4}{5}. \tag{3.26}$$

Since the sequence $\{D_n\}_{n\geq 2}$ is log-convex, that is,

$$\frac{1}{D_{n+1}^2} > \frac{1}{D_{n+2}D_n}, \tag{3.27}$$

applying (3.26) to the righthand side of (3.24), we deduce that (3.21) holds for $n \geq 5$. On the other hand, it is easy to verify that (3.21) holds for $2 \leq n \leq 5$. This proves that the sequence $\{D_n\}_{n\geq 2}$ is ratio log-concave. This completes the proof. ∎

Since $\sqrt[4]{D_4}/\sqrt[3]{D_3} > \sqrt[5]{D_5}/\sqrt[4]{D_4}$, by Theorem 3.1 we deduce the known result that the sequence $\{\sqrt[n]{D_n}\}_{n\geq 3}$ is strictly log-concave.

The above approach also applies to ratio log-convex sequences. We have the following criterion.

Theorem 3.6 *If a sequence $\{a_n\}_{n\geq k}$ is ratio log-convex and*

$$\frac{\sqrt[k+1]{a_{k+1}}}{\sqrt[k]{a_k}} < \frac{\sqrt[k+2]{a_{k+2}}}{\sqrt[k+1]{a_{k+1}}},$$

then the sequence $\{\sqrt[n]{a_n}\}_{n\geq k}$ is strictly log-convex.

As an application, we consider the generalized harmonic numbers. For any positive integers m, n, the n-th generalized harmonic number $H_{n,m}$ of order m is defined by

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}. \tag{3.28}$$

Theorem 3.7 *For any positive integer m, the sequence $\{H_{n,m}\}_{n\geq 1}$ is ratio log-convex.*

Proof. To prove that the sequence $\{H_{n,m}\}_{n\geq 1}$ is ratio log-convex, we aim to show that $\frac{H_{n+2,m}H_{n,m}}{H^2_{n+1,m}}$ is strictly increasing in n. From (3.28), we get the following recurrence relations

$$H_{n+2,m} = H_{n+1,m} + \frac{1}{(n+2)^k} \tag{3.29}$$

and

$$H_{n,m} = H_{n+1,m} - \frac{1}{(n+1)^k}. \tag{3.30}$$

It follows that

$$\begin{aligned} \frac{H_{n+2,m}H_{n,m}}{H^2_{n+1,m}} &= 1 - \left(\frac{1}{(n+1)^m} - \frac{1}{(n+2)^m}\right)\frac{1}{H_{n+1,m}} \\ &\quad - \frac{1}{(n+1)^m(n+2)^m H^2_{n+1,m}}. \end{aligned} \tag{3.31}$$

Since for $x > 0$, $(x^{-m} - (x+1)^{-m})$ is strictly decreasing, we see that $\frac{1}{n^m} - \frac{1}{(n+1)^m}$ is strictly decreasing in n. Thus from (3.31), we deduce that $\frac{H_{n+2,m}H_{n,m}}{H_{n+1,m}^2}$ is strictly increasing in n. This implies that the sequence $\{H_{n,m}\}_{n\geq 1}$ is ratio log-convex. This completes the proof. ∎

Sun [23] conjectured that the sequence $\{\sqrt[n]{H_{n,m}}\}_{n\geq 3}$ is strictly log-convex. This has been proved by Hou, Sun and Wen [15]. Since for any positive integer m, $\sqrt[4]{H_{4,m}}/\sqrt[3]{H_{3,m}} < \sqrt[5]{H_{5,m}}/\sqrt[4]{H_{4,m}}$. Thus by Theorem 3.6 and Theorem 3.7, we are led to the above result.

4 Sequences with Three-term Recurrences

In this section, we consider the sequences $\{a_n\}_{n\geq 0}$ with a three-term recurrence relation

$$a_n = u(n)a_{n-1} + v(n)a_{n-2} \tag{4.1}$$

where $u(n)$ and $v(n)$ are rational functions and for $n \geq 2$, $u(n) > 0$. We shall give sufficient conditions for $\{a_n\}_{n\geq k}$ to be ratio log-concave. In this way, we prove that the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the numbers of the treelike polyhexes and the Domb numbers are ratio log-concave. As consequences, we obtain the results of the Luca and Stănică on the Motzkin numbers and the central Delannoy numbers. This also leads to a confirmation of a conjecture of Sun [23] on the log-behavior of the Domb numbers.

First, we consider the case when $v(n) < 0$ for any $n \geq 2$.

Theorem 4.1 *Let $\{a_n\}_{n\geq 0}$ be the sequence defined by the recurrence relation (4.1). Assume that for $n \geq 2$, $v(n) > 0$ and $u(n)^3 > u(n+1)v(n)$. If there exists a positive integer N and a function $g(n)$ such that for all $n \geq N$,*

(i) $a_n/a_{n-1} \geq g(n) \geq u(n)$;

(ii) $g(n)^4 - u(n)g(n)^3 - u(n+1)v(n)g(n) - v(n)v(n+1) > 0$,

then the sequence $\{a_n\}_{n\geq N-2}$ is ratio log-concave.

Proof. To prove that the sequence $\{a_n\}_{n\geq N-2}$ is ratio log-concave, we proceed to show that for $n \geq N$,

$$a_n^3 a_{n-2} - a_{n+1}a_{n-1}^3 > 0. \tag{4.2}$$

By the recurrence relation (4.1), we deduce that

$$
\begin{aligned}
& a_n^3 a_{n-2} - a_{n+1} a_{n-1}^3 \\
= \ & \frac{1}{v(n)} a_n^3 (a_n - u(n) a_{n-1}) - (u(n+1) a_n + v(n+1) a_{n-1}) a_{n-1}^3 \\
= \ & \frac{a_{n-1}^4}{v(n)} \left[\left(\frac{a_n}{a_{n-1}} \right)^4 - u(n) \left(\frac{a_n}{a_{n-1}} \right)^3 - u(n+1) v(n) \left(\frac{a_n}{a_{n-1}} \right) - v(n) v(n+1) \right].
\end{aligned}
$$

Since $v(n) > 0$ for $n \geq 2$, in order to prove (4.2), it suffices to show that for $n \geq N$,

$$
\left(\frac{a_n}{a_{n-1}} \right)^4 - u(n) \left(\frac{a_n}{a_{n-1}} \right)^3 - u(n+1) v(n) \left(\frac{a_n}{a_{n-1}} \right) - v(n) v(n+1) > 0. \quad (4.3)
$$

Define

$$
f(x) = x^4 - u(n) x^3 - u(n+1) v(n) x - v(n) v(n+1).
$$

Then (4.3) can be written as $f(\frac{a_n}{a_{n-1}}) > 0$. Noting that

$$
f'(x) = 4x^3 - 3u(n) x^2 - u(n+1) v(n)
$$

and

$$
f''(x) = 12x^2 - 6u(n) x.
$$

we have $f''(x) > 0$ for $x > u(n)/2$. This implies that for $x > u(n)/2$, $f'(x)$ is strictly increasing. Since $u(n)^3 > u(n+1) v(n)$, we have $f'(u(n)) > 0$. It follows that for $x \geq u(n)$, $f'(x) > 0$. Thus $f(x)$ is increasing for $x \geq u(n)$. Since $g(n) \geq u(n)$, we deduce that $f(x)$ is strictly increasing for $x \geq g(n)$. On the other hand, condition (ii) in the theorem asserts that $f(g(n)) > 0$ for any $n \geq N$. Thus $f(x) > 0$ for $x \geq g(n)$. Since $a_n / a_{n-1} \geq g(n)$, we have $f(\frac{a_n}{a_{n-1}}) > 0$. This completes the proof. ∎

To apply the above theorem, we need a lower bound $g(n)$ on the ratio a_n / a_{n-1} subject to conditions (i) and (ii) in Theorem 4.1. The following lemma shows how to find such a lower bound.

Lemma 4.2 *Let $\{a_n\}_{n \geq 0}$ be the sequence defined by the recurrence relation (4.1). Assume that $v(n) > 0$ for $n \geq 2$. If there exists a positive integer N and a function $g(n)$ such that*

$$
g(N) < \frac{a_N}{a_{N-1}} < \frac{v(N+1)}{g(N+1) - u(N+1)}
$$

and the inequality

$$
u(n) + \frac{v(n)}{g(n-1)} < \frac{v(n+1)}{g(n+1) - u(n+1)} \quad (4.4)
$$

holds for all $n \geq N$, then for $n \geq N$,

$$g(n) < \frac{a_n}{a_{n-1}} < \frac{v(n+1)}{g(n+1) - u(n+1)}. \tag{4.5}$$

Proof. We use induction on n. Assume that for $n = m \geq N$,

$$g(m) < \frac{a_m}{a_{m-1}} < \frac{v(m+1)}{g(m+1) - u(m+1)}. \tag{4.6}$$

We proceed to show that (4.6) holds for $n = m+1$, that is,

$$g(m+1) < \frac{a_{m+1}}{a_m} < \frac{v(m+2)}{g(m+2) - u(m+2)}. \tag{4.7}$$

By recurrence relation (4.1), we find that

$$\frac{a_{m+1}}{a_m} = \frac{u(m+1)a_m + v(m+1)a_{m-1}}{a_m} = u(m+1) + v(m+1)\frac{a_{m-1}}{a_m}.$$

So we need to show that

$$g(m+1) < u(m+1) + v(m+1)\frac{a_{m-1}}{a_m} < \frac{v(m+2)}{g(m+2) - u(m+2)}. \tag{4.8}$$

Note that the induction hypothesis

$$\frac{a_m}{a_{m-1}} < \frac{v(m+1)}{g(m+1) - u(m+1)}$$

can be rewritten as

$$u(m+1) + v(m+1)\frac{a_{m-1}}{a_m} > g(m+1), \tag{4.9}$$

which proves the first inequality in (4.8). On the other hand, by condition (4.4) with $n = m+1$, we get

$$u(m+1) + \frac{v(m+1)}{g(m)} < \frac{v(m+2)}{g(m+2) - u(m+2)}, \tag{4.10}$$

By the induction hypothesis $g(m) < a_m/a_{m-1}$ and inequality (4.10), we find that

$$u(m+1) + v(m+1)\frac{a_{m-1}}{a_m} < u(m+1) + \frac{v(m+1)}{g(m)} < \frac{v(m+2)}{g(m+2) - u(m+2)}. \tag{4.11}$$

Hence the second inequality in (4.8) holds. This completes the proof. ∎

Notice that Theorem 4.1 and Lemma 4.2 involve a lower bound $g(n)$ of a_n/a_{n-1} subject to condition (ii) in Theorem 4.1 and the inequality (4.4).

Following the idea in [9], we give a heuristic approach to finding such a lower bound. It should be noted that the success of the following process is not guaranteed. But it serves the purpose in many cases.

Let us begin with the quadratic equation

$$\lambda^2 - u(n)\lambda - v(n) = 0. \tag{4.12}$$

Since $v(n) > 0$ for $n \geq 2$, this equation has a unique positive root

$$\lambda(n) = \frac{u(n) + \sqrt{u(n)^2 + 4v(n)}}{2}. \tag{4.13}$$

We take $g(n) = \lambda(n)$. If $g(n)$ satisfies condition (ii) in Theorem 4.1 and (4.4), then it is a feasible choice. Otherwise, we try to get a function $s(n)$ such that $s(n) > \lambda(n)$ for any nonnegative integer n. Since $u(n)$ and $v(n)$ are rational functions, assume may that

$$u(n)^2 + 4v(n) = \frac{P(n)}{Q(n)}, \tag{4.14}$$

where $P(n)$ and $Q(n)$ are polynomials in n. If $P(n)$ can be written as $R(n)^2 - c$, where $R(n)$ is a polynomial in n and c is a positive number, then we have

$$\sqrt{u(n)^2 + 4v(n)} < \frac{R(n)}{\sqrt{Q(n)}}.$$

Thus $s(n)$ can be chosen as follows

$$s(n) = \frac{u(n)\sqrt{Q(n)} + R(n)}{2\sqrt{Q(n)}}. \tag{4.15}$$

Again, take $g(n) = s(n)$. If $g(n)$ satisfies condition (ii) in Theorem 4.1 and (4.4), then it is the desired lower bound. Otherwise, we try to find a number x such that

$$g(n) = s(n) + \frac{1}{d(n)}\frac{x}{n} \tag{4.16}$$

satisfies condition (ii) in Theorem 4.1 and (4.4), where $d(n)$ is the denominator of $s(n)$. Since the lower bound $g(n)$ in Lemma 4.2 satisfies the two inequalities in (4.5), this implies that

$$\frac{v(n+1)}{g(n+1) - u(n+1)} > g(n). \tag{4.17}$$

We shall be guided by the above inequality (4.17) to find the number x. More precisely, let

$$C(x, n) = \frac{v(n+1)}{g(n+1) - u(n+1)} - g(n), \tag{4.18}$$

where $g(n)$ is given by (4.16), and

$$g(n+1) = s(n+1) + \frac{1}{d(n+1)} \frac{x}{n+1}.$$

If $C(x, n)$ is a rational function, then let

$$C(x, n) = \frac{Y(x, n)}{Z(x, n)},$$

where $Y(x, n)$ and $Z(x, n)$ are polynomials in x and n. To find x, we treat $Y(x, n)$ as a polynomial $Y(n)$ in n. Denote by $H(x)$ the coefficient of the term of the highest degree in $Y(n)$, and set $H(x) = 0$. If x_1 is a solution of $H(x) = 0$, then we set

$$g(n) = s(n) + \frac{1}{d(n)} \frac{x_1}{n}.$$

If $g(n)$ satisfies condition (ii) in Theorem 4.1 and (4.4), then it is the desired lower bound. Otherwise, we repeat the above process to find a number x_2 such that

$$g(n) = s(n) + \frac{1}{d(n)} \left(\frac{x_1}{n} + \frac{x_2}{n^2} \right)$$

satisfies condition (ii) in Theorem 4.1 and (4.4). If we are lucky, by iteration we may find numbers $x_1, x_2, \ldots, x_k$ such that

$$g(n) = s(n) + \frac{1}{d(n)} \left(\frac{x_1}{n} + \frac{x_2}{n^2} + \cdots + \frac{x_k}{n^k} \right)$$

satisfies the lower bound condition (ii) in Theorem 4.1 and (4.4). This leads to a lower bound $g(n)$ of the ratio a_n/a_{n-1}.

For example, we consider the Motzkin numbers M_n defined by the recurrence relation

$$M_n = \frac{2n+1}{n+2} M_{n-1} + \frac{3n-3}{n+2} M_{n-2}, \tag{4.19}$$

where $n \geq 2$ and $M_0 = M_1 = 1$, see Aigner [1]. Let $u(n) = (2n+1)/(n+2)$ and $v(n) = (3n-3)/(n+2)$. It is easy to see that for $n \geq 2$, $v(n) > 0$ and $u(n)^3 > u(n+1)v(n)$, that is, $v(n)$ and $u(n)$ satisfy conditions in Theorem 4.1. Since equation (4.12) has a unique positive root

$$\lambda(n) = \frac{2n+1+\sqrt{16n^2+16n-23}}{2(n+2)}.$$

Clearly, $\lambda(n)$ does not satisfy condition (ii) in Theorem 4.1. Thus we continue to find a function $s(n)$ such that $s(n) < \lambda(n)$ for any positive integer n. Since

$$u(n)^2 + 4v(n) = \frac{(4n+2)^2 - 27}{(n+2)^2},$$

we have $R(n) = 4n + 2$ and $Q(n) = (n+2)^2$. Thus, by the definition (4.15) of $s(n)$, we have

$$s(n) = \frac{3n + \frac{3}{2}}{n+2}$$

It is easy to see that $s(n)$ does not satisfy inequality (4.4) in Lemma 4.2. By the definition (4.18) of $C(x, n)$, we have

$$C(x,n) = \frac{-(16x+9)n^2 - (16x+9)n - 4x^2 - 6x}{2(2n^2+5n+3+2x)(n+2)n}.$$

Let $Y(x, n)$ denote the numerator of $C(x, n)$, which is a polynomial of degree 2 in n. Setting the coefficient of n^2 in $Y(x, n)$ to zero, we get $-9 - 16x = 0$ and $x_1 = -\frac{9}{16}$. Then let

$$g(n) = \frac{6n^2 + 3n - \frac{9}{8}}{2n(n+2)}.$$

It is easy to verify that for $n \geq 13$, $g(n)$ satisfies the conditions in Theorem 4.1 and Lemma 4.2. Thus we deduce that $\{M_n\}_{n\geq 11}$ is ratio log-concave. Moreover, it can be checked that for $6 \leq n \leq 12$,

$$M_n^3 M_{n-2} > M_{n+1} M_{n-1}^3.$$

Hence we arrive at the following assertion.

Theorem 4.3 *The sequence $\{M_n\}_{n\geq 4}$ is ratio log-concave.*

Sun [23] conjectured that the sequence $\{\sqrt[n]{M_n}\}_{n\geq 1}$ is strictly log-concave. This has been proved by Luca and Stănică [19]. Since for $2 \leq n \leq 5$, $(\sqrt[n]{M_n})^2 > \sqrt[n+1]{M_{n+1}}\sqrt[n-1]{M_{n-1}}$, this fact is also a consequence of Theorem 3.1 and Theorem 4.3.

The Fine numbers f_n are given by the recurrence relation

$$2(n+1)f_n = (7n-5)f_{n-1} + 2(2n-1)f_{n-2}, \tag{4.20}$$

where $n \geq 2$ and $f_0 = 1$ and $f_1 = 0$, see Deutsch and Shapiro [11]. Next we show the sequence $\{f_n\}_{n\geq 5}$ is ratio log-concave.

Theorem 4.4 *The sequence $\{f_n\}_{n\geq 5}$ is ratio log-concave, and the sequence $\{\sqrt[n]{f_n}\}_{n\geq 2}$ is strictly log-concave.*

Proof. Let $u(n) = (7n-5)/(2n+2)$ and $v(n) = (2n-1)/(n+1)$. Clearly, for $n \geq 2$ we have $v(n) > 0$ and $u(n)^3 > u(n+1)v(n)$, that is, $v(n)$ and $u(n)$

satisfy the conditions in Theorem 4.1. Using the above procedure to find a lower bound $g(n)$ of the ratio f_n/f_{n-1}, we get

$$g(n) = \frac{4n^2 - 2n + \frac{2}{3}}{n^2 + n}.$$

It is easy to verify that for $n \geq 7$, $g(n)$ satisfies conditions in Theorem 4.1 and Lemma 4.2. Thus by Theorem 4.1, we deduce that $\{f_n\}_{n\geq 5}$ is ratio log-concave. Moreover, for $2 \leq n \leq 5$ we have

$$(\sqrt[n]{f_n})^2 > \sqrt[n+1]{f_{n+1}}\,\sqrt[n-1]{f_{n-1}}.$$

Hence, we conclude that $\{\sqrt[n]{f_n}\}_{n\geq 1}$ is strictly log-concave. ∎

For the case when $v(n) < 0$ for any $n \geq 2$, using the argument in the proof of Theorem 4.1, we deduce the following criterion.

Theorem 4.5 *Let $\{a_n\}_{n\geq 0}$ be the sequence defined by the recurrence relation (4.1). Assume that $v(n) < 0$ for $n \geq 2$. If there exists a positive integer N and a function $h(n)$ such that for all $n \geq N$,*

(i) $3u(n)/4 \leq a_n/a_{n-1} \leq h(n)$;

(ii) $h(n)^4 - u(n)h(n)^3 - u(n+1)v(n)h(n) - v(n)v(n+1) < 0$,

then $\{a_n\}_{n\geq N-2}$ is ratio log-concave.

To apply the above theorem, we need an upper bound $h(n)$ of the ratio a_n/a_{n-1} subject to conditions (i) and (ii). The following lemma can be used to derive such an upper bound. The proof of this lemma is similar to that of Lemma 4.2, and hence it is omitted.

Lemma 4.6 *Let $\{a_n\}_{n\geq 0}$ be the sequence defined by the recurrence relation (4.1). Assume that $v(n) < 0$ for $n \geq 2$. If there exists a positive integer N and a function $h(n)$ such that $a_N/a_{N-1} < h(N)$ and the inequality*

$$h(n+1) > u(n+1) + \frac{v(n+1)}{h(n)}, \tag{4.21}$$

holds for all $n \geq N$, then for $n \geq N$, $a_n/a_{n-1} < h(n)$.

We now describe a heurestic procedure to find a rational function $h(n)$ satisfying condition (ii) in Theorem 4.5 and inequality (4.21) in Lemma 4.6. Since $v(n) < 0$ for any $n \geq 2$, equation (4.12) either has no real root or two positive roots. If equation (4.12) has two positive roots, then $\lambda(n)$ given by

(4.13) is the larger root. Since we need an upper bound $h(n)$ of $a_n/a_{n=1}$, we take $h(n) = \lambda(n)$. If $h(n)$ satisfies condition (ii) in Theorem 4.5 and (4.21), then it is a feasible choice. Otherwise, we try to get a function $r(n)$ such that $r(n) < \lambda(n)$ for any nonnegative integer n. If $P(n)$ given by (4.14) can be written as $T(n)^2 + c$, where $T(n)$ is a polynomial in n and c is a positive number, then $r(n)$ can be chosen as follows

$$r(n) = \frac{u(n)\sqrt{Q(n)} + T(n)}{2\sqrt{Q(n)}}, \tag{4.22}$$

where $Q(n)$ is given by (4.14). Again, take $h(n) = r(n)$. If $h(n)$ satisfies condition (ii) in Theorem 4.5 and (4.21), then it is a desired upper bound. Otherwise, we try to find a number x such that

$$h(n) = r(n) + \frac{1}{d(n)}\frac{x}{n} \tag{4.23}$$

satisfies condition (ii) in Theorem 4.5 and (4.21), where $d(n)$ is the denominator of $r(n)$. We use inequality (4.21) to find the number x. Let

$$D(x,n) = h(n+1) - u(n+1) - \frac{v(n+1)}{h(n)}, \tag{4.24}$$

where $h(n)$ is given by (4.23), and

$$h(n+1) = r(n+1) + \frac{1}{d(n+1)}\frac{x}{n+1}.$$

If $D(x,n)$ is a rational function, then we obtain x_1 from $D(x,n)$ in the same manner as we get x_1 from $C(x,n)$ in the aforementioned procedure of deriving $g(n)$ as a lower bound of a_n/a_{n-1}. We set

$$h(n) = r(n) + \frac{1}{d(n)}\frac{x_1}{n}.$$

If $h(n)$ satisfies condition (ii) in Theorem 4.5 and (4.21), then it is a feasible upper bound. Otherwise, we repeat the above procedure to find a number x_2 such that

$$h(n) = r(n) + \frac{1}{d(n)}\left(\frac{x_1}{n} + \frac{x_2}{n^2}\right)$$

satisfies condition (ii) in Theorem 4.5 and (4.21). Eventually, by using this process we may find numbers $x_1,\ x_2,\ \ldots,\ x_k$ such that

$$h(n) = r(n) + \frac{1}{d(n)}\left(\frac{x_1}{n} + \frac{x_2}{n^2} + \cdots + \frac{x_k}{n^k}\right)$$

satisfies the upper bound condition (ii) in Theorem 4.5 and (4.21). Then we get an upper bound $h(n)$ of the ratio a_n/a_{n-1}.

For example, let us consider the central Delannoy numbers $D(n)$ defined by the recurrence relation

$$D(n) = \frac{3(2n-1)}{n} D(n-1) - \frac{n-1}{n} D(n-2), \tag{4.25}$$

where $n \geq 2$ and $D(0) = 1$ and $D(1) = 3$, see Sun [22].

Let $u(n) = 3(2n-1)/n$ and $v(n) = -(n-1)/n$. Clearly, for $n \geq 2$ we have $v(n) < 0$ and $D(n)/D(n-1) \geq 3u(n)/4$, that is, $u(n)$ and $v(n)$ meet the requirements of Theorem 4.5. Since the larger root of equation (4.12) is

$$\lambda(n) = \frac{6n - 3 + \sqrt{32n^2 - 32n + 9}}{2n},$$

It is easy to verify that $\lambda(n)$ does not satisfy condition (ii) in Theorem 4.5. We continue to find a function $r(n)$ such that $r(n) < \lambda(n)$ for any positive integer n. Since

$$u(n)^2 + 4v(n) = \frac{2(4n-2)^2 + 1}{n^2},$$

we take $T(n) = 4\sqrt{2}n - 2\sqrt{2}$ and $Q(n) = n^2$. Thus, by the definition (4.22) of $r(n)$, we have

$$r(n) = \frac{(3 + 2\sqrt{2})n - \frac{3}{2} - \sqrt{2}}{n}.$$

It can be checked that $r(n)$ does not satisfy condition (ii) in Theorem 4.5. So we further consider

$$h(n) = \frac{(3 + 2\sqrt{2})n - \frac{3}{2} - \sqrt{2}}{n} + \frac{1}{n}\frac{x}{n}.$$

By the definition (4.24) of $D(x, n)$, we have

$$D(x, n) = \frac{(16\sqrt{2}x + 1)n^2 - (24x - 8\sqrt{2}x - 1)n + 4\sqrt{2}x - 6x + 4x^2}{2(n+1)(6n^2 + 4\sqrt{2}n^2 - 2\sqrt{2}n - 3n + 2x)}.$$

Let $Y(x, n)$ be the numerator of $D(x, n)$, which is a polynomial of degree 2 in n. Setting the coefficient of n^2 in $Y(x, n)$ to zero, we obtain $16\sqrt{2}x + 1 = 0$, $x_1 = -\sqrt{2}/32$ and

$$h(n) = \frac{(3 + 2\sqrt{2})n^2 - \frac{3}{2}n - \sqrt{2}n - \frac{\sqrt{2}}{32}}{n^2}.$$

Now, for $n \geq 2$, $h(n)$ satisfies the conditions in Theorem 4.5 and Lemma 4.6. By Theorem 4.5, we come to the following conclusion.

Theorem 4.7 *The sequence $\{D(n)\}_{n\geq 0}$ is ratio log-concave.*

Note that Sun [23] conjectured that the sequence $\{\sqrt[n]{D(n)}\}_{n\geq 1}$ is strictly log-concave. This has been proved by Luca and Stănică [19]. Since $D(2) > D(1)\sqrt[3]{D(3)}$, this property also follows from the ratio log-concavity of the sequence $\{D(n)\}_{n\geq 0}$.

Let t_n be the number of tree-like polyhexes with $n+1$ hexagons, which is given by the recurrence relation

$$(n+1)t_n = 3(2n-1)t_{n-1} - 5(n-2)t_{n-2},$$

where $n \geq 2$ and $t_0 = t_1 = 1$, see Harary and Read [14]. Next we prove the ratio log-concavity of the sequence $\{t_n\}_{n\geq 0}$.

Theorem 4.8 *The sequence $\{t_n\}_{n\geq 0}$ is ratio log-concave, and the sequence $\{\sqrt[n]{t_n}\}_{n\geq 1}$ is strictly log-concave.*

Proof. Let $u(n) = (6n-3)/(n+1)$ and $v(n) = -(5n-10)/(n+1)$. Clearly, for $n \geq 2$ we have $v(n) < 0$ and $t_n/t_{n-1} \geq 3u(n)/4$, that is, $u(n)$ and $v(n)$ satisfy the conditions in Theorem 4.5. Using the above procedure to find an upper bound $h(n)$ of the ratio t_n/t_{n-1}, we get

$$h(n) = \frac{10n^3 - 5n^2 + \frac{15}{8}n + 6}{2n^2 + 2n^3}.$$

For $n \geq 7$, it can be verified that $h(n)$ satisfies the conditions in Theorem 4.5 and Lemma 4.6. Thus by Theorem 4.5, we deduce that $\{t_n\}_{n\geq 5}$ is ratio log-concave. Moreover, for $2 \leq n \leq 6$, we have

$$t_n^3 t_{n-2} - t_{n+1} t_{n-1}^3 > 0.$$

Thus the sequence $\{t_n\}_{n\geq 0}$ is ratio log-concave.

Since $\sqrt{t_2}/t_1 > \sqrt[3]{t_3}/\sqrt{t_2}$, by Theorem 3.1 we conclude that the sequence $\{\sqrt[n]{t_n}\}_{n\geq 1}$ is strictly log-concave. ∎

Finally, we consider the sequence of the Domb numbers D_n given by the recurrence relation

$$n^3 D_n = 2(2n-1)(5n^2 - 5n + 2)D_{n-1} - 64(n-1)^3 D_{n-2},$$

where $n \geq 2$ and $D_0 = 1$ and $D_1 = 4$. The n-th Domb number D_n is the number of $2n$-step polygons on the diamond lattice. Chan, Chan and Liu [7] obtained a series for $\frac{1}{\pi}$ involving the Domb numbers. Chan, Tanigawa, Yangc, and Zudilin [8] found three analogues of Clausen's identities involving Domb numbers. Sun [23] conjectured that the sequence $\{D_n\}_{n\geq 0}$ is log-convex and

the sequence $\{\sqrt[n]{D_n}\}_{n\geq 1}$ is strictly increasing and strictly log-concave. Wang and Zhu [21] proved the sequence $\{D_n\}_{n\geq 0}$ is log-convex and the sequence $\{\sqrt[n]{D_n}\}_{n\geq 1}$ is increasing. Next we show $\{D_n\}_{n\geq 0}$ is ratio log-concave. As a consequence, we confirm the conjecture of Sun on the strict log-concavity of $\{\sqrt[n]{D_n}\}_{n\geq 1}$.

While we shall follow the procedure to derive an upper bound $h(n)$ of D_n/D_{n-1}, we observe that by doing a slight adjustment of the first estimate we may obtain a feasible upper bound without iteration.

Theorem 4.9 *The sequence $\{D_n\}_{n\geq 0}$ is ratio log-concave, and the sequence $\{\sqrt[n]{D_n}\}_{n\geq 1}$ is strictly log-concave.*

Proof. Let $u(n) = 2(2n-1)(5n^2-5n+2)/n^3$ and $v(n) = -64(n-1)^3/n^3$. It is easy to verify that for $n \geq 2$, $v_n < 0$ and for $n \geq 24$, $D_n/D_{n-1} > 3u(n)/4$, that is, $u(n)$ and $v(n)$ satisfy the conditions in Theorem 4.5. By the definition (4.13) of $\lambda(n)$, we have

$$\begin{aligned}\lambda(n) &= \frac{20n^3-30n^2+18n-4}{2n^3} \\ &\quad + \frac{\sqrt{(12n^3-18n^2+22n-8)^2+208n^2-208n+48}}{2n^3}.\end{aligned}$$

It can be verified that $\lambda(n)$ does not satisfy condition (ii) in Theorem 4.5. In order to find a rational function $r(n)$ such that $r(n) < \lambda(n)$ for $n \geq 0$, we may drop off $208n^2 - 208n + 48$ in the square root. Now we set

$$r(n) = \frac{16n^3-24n^2+40n-12}{n^3}.$$

Notice that $r(n)$ does not satisfy condition (ii) in Theorem 4.5. By adjusting the constant term and the coefficient of n in $r(n)$, we get the rational function

$$h(n) = \frac{16n^3-24n^2+12n-2}{n^3}.$$

It is clear that $h(n) < r(n)$ for $n \geq 2$. It can be checked that for $n \geq 24$, $h(n)$ satisfies the conditions in Theorem 4.5 and Lemma 4.6. Thus by Theorem 4.5 the sequence $\{D_n\}_{n\geq 22}$ is ratio log-concave. Moreover, for $2 \leq n \leq 23$ $D_n^3 D_{n-2} > D_{n-1}^3 D_{n+1}$. Thus the sequence $\{D_n\}_{n\geq 0}$ is ratio log-concave.

Since $\sqrt{D_2}/D_1 > \sqrt[3]{D_3}/\sqrt{D_2}$, by Theorem 3.1, we conclude that the sequence $\{\sqrt[n]{D_n}\}_{n\geq 1}$ is strictly log-concave, which confirms the conjecture of Sun [23]. ∎

To conclude, we conjecture that the sequences of the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the numbers of treelike

polyhexes and the Domb numbers are almost infinitely logarithmically monotonic. More precisely, we say that a sequence is almost infinitely logarithmically monotonic if for each $k \geq 0$, it is logarithmically monotonic of order k except for certain terms at the beginning.

We also conjecture that the sequence of the Bell numbers B_n is almost infinitely logarithmically monotonic, where B_n is the number of partitions of $\{1, 2, \ldots, n\}$.

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